Partial Fraction

Consider

$$F(s) = \frac{N(s)}{D(s)} \tag{1}$$

where $N(s)$ and $D(s)$ are polynomials in s, $D(s)$ being of higher order than $N(s)$, i.e.,

$$\lim_{s\to\infty} F(s) = 0 \tag{2}$$

1. The characteristic polynomial $D(s)$ has distinct roots (real or complex), i.e.,

$$D(s) = (s - r_1)(s - r_2)\cdots(s - r_n) = \prod_{i=1}^{n}(s - r_i) \tag{3}$$

Let

$$F(s) = \frac{\alpha_1}{s - r_1} + \frac{\alpha_2}{s - r_2} + \cdots + \frac{\alpha_n}{s - r_n} \tag{4}$$

Then it is obvious that

$$\lim_{s\to r_i}(s - r_i)F(s) = \alpha_i \tag{5}$$

for $i = 1, 2, \cdots, n$. Therefore,

$$\begin{aligned}\alpha_i &= \lim_{s\to r_i}\frac{(s - r_i)N(s)}{D(s)} \\ &= N(r_i)\lim_{s\to r_i}\frac{s - r_i}{D(s)} \\ &= N(r_i)\frac{\frac{d}{ds}(s - r_i)|_{s=r_i}}{D'(r_i)} \\ \Rightarrow \qquad \alpha_i &= \frac{N(r_i)}{D'(r_i)} \end{aligned} \tag{6}$$

Substituting into (4), one obtain

$$f(t) = \mathcal{L}^{-1}\left[\sum_{i=1}^{n}\frac{N(r_i)}{D'(r_i)}\frac{1}{s - r_i}\right] = \sum_{i=1}^{n}\frac{N(r_i)}{D'(r_i)}e^{r_i t} \tag{7}$$

[Example]

$$F(s) = \frac{1}{(s+1)(s^2+1)}$$

$$D(s) = s^3 + s^2 + s + 1$$
$$D'(s) = 3s^2 + 2s + 1$$

The roots of $D(s) = 0$ are -1, $j = \sqrt{-1}$ and $-j$.

i	r_i	$N(r_i)$	$D'(r_i)$	α_i
1	-1	1	2	$1/2$
2	j	1	$2(-1+j)$	$\frac{1}{2(-1+j)}$
3	$-j$	1	$2(-1-j)$	$\frac{1}{2(-1-j)}$

$$\begin{aligned} f(t) &= \frac{1}{2}e^{-t} + \frac{1}{2(-1+j)}e^{jt} + \frac{1}{2(-1-j)}e^{-jt} \\ &= \frac{1}{2}\left[e^{-t} - \frac{1}{2}(e^{jt}+e^{-jt}) - \frac{j}{2}(e^{jt}-e^{-jt})\right] \\ &= \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right) \end{aligned}$$

2. $D(s)$ contains repeating roots.

$$D(s) = (s-r_1)^m (s-r_{m+1})(s-r_{m+2}) \cdots (s-r_n) \tag{8}$$

$$F(s) = \frac{\alpha_1}{(s-r_1)^m} + \frac{\alpha_2}{(s-r_1)^{m-1}} + \cdots + \frac{\alpha_m}{s-r_1} + \frac{\alpha_{m+1}}{s-r_{m+1}} + \cdots + \frac{\alpha_n}{s-r_n} \tag{9}$$

The coefficients $\alpha_{m+1}, \alpha_{m+2}, \cdots, \alpha_n$ can be obtained by the same method described previously. The remaining m coefficients can be determined as follows:

$$F(s) = \frac{\alpha_1}{(s-r_1)^m} + \frac{\alpha_2}{(s-r_1)^{m-1}} + \cdots + \frac{\alpha_m}{s-r_1} + h(s) \tag{10}$$

Multiply both side by $(s-r_1)^m$

$$(s-r_1)^m F(s) = \alpha_1 + (s-r_1)\alpha_2 + \cdots + (s-r_1)^{m-1}\alpha_m + h(s)(s-r_1)^m \tag{11}$$

Differentiate both sides $(i-1)$ times w.r.t. s $(i \leq m)$

$$\frac{d^{i-1}}{ds^{i-1}}\left[(s-r_1)^m F(s)\right] = (i-1)!\alpha_i + \frac{i!}{1}\alpha_{i+1}(s-r_1) + \frac{(i+1)!}{2!}\alpha_{i+2}(s-r_1)^2 + \cdots$$
$$+\frac{(m-1)!}{(m-i)!}\alpha_m(s-r_1)^{m-i} + \frac{d^{i-1}}{ds^{i-1}}\left[(s-r_1)^m h(s)\right] \tag{12}$$

Thus,

$$\lim_{s\to r_1}\frac{d^{i-1}}{ds^{i-1}}\left[(s-r_1)^m F(s)\right] = (i-1)!\alpha_i \tag{13}$$

and

$$\alpha_i = \frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[(s-r_1)^m F(s)\right]_{s=r_1} \tag{14}$$

The inversion of $F(s)$ can be written as

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\sum_{i=1}^{m}\alpha_i\frac{1}{(s-r_1)^{m-i+1}} + \sum_{i=m+1}^{n}\alpha_i\frac{1}{s-r_i}\right] \tag{15}$$

Since

$$\mathcal{L}\left\{\frac{1}{(n-1)!}t^{n-1}e^{at}\right\} = \frac{1}{(s-a)^n} \tag{16}$$

The result of inversion is

$$f(t) = \sum_{i=1}^{m}\alpha_i\frac{t^{m-i}}{(m-i)!}e^{r_1 t} + \sum_{i=m+1}^{n}\alpha_i e^{r_i t} \tag{17}$$

[**Exercise**]

$$F(s) = \frac{1}{s(s+1)(s+2)^3}$$

Ans:

$$f(t) = e^{-2t}\left[\frac{t^2}{4} + \frac{3t}{4} + \frac{7}{8}\right] - e^{-t} + \frac{1}{8}$$